Fixed points of holomorphic transformations of operator balls

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Abstract. A new technique for proving fixed point theorems for families of holomorphic transformations of operator balls is developed. One of these theorems is used to show that a bounded representation in a real or complex Hilbert space is orthogonalizable or unitarizable (that is similar to an orthogonal or unitary representation), respectively, provided the representation has an invariant indefinite quadratic form with finitely many negative squares.

1   Introduction

The main purpose of this paper is to prove the existence of fixed points for some collections of holomorphic maps of the unit ball $\mathcal{B}$ in the space $L(K, H)$ of operators from a finite-dimensional Hilbert space K to a Hilbert space H. More precisely we consider 1) groups of biholomorphic maps, and 2) finite commutative families of holomorphic maps, and prove that in both cases the existence of a fixed point is equivalent to the existence of an invariant set separated from the boundary of $\mathcal{B}$. For the first case, this result was obtained in [17]; the advantage of the present approach is that we do not use the deep theory of Shafrir [23] replacing it by some simple general observations from metric geometry.

    The standard tool in the study of holomorphic maps of $\mathcal{B}$ is the Carathéodory metric $\rho = c_{\mathcal{B}}$: it is known that holomorphic maps do not increase $c_{\mathcal{B}}$. A similar metric ρ can be defined in the case where Hilbert spaces K and H are over reals. We treat both cases simultaneously dealing with ρ-non-expansive maps. This allows us to obtain applications

to invariant subspaces and the orthogonalization problem in real Hilbert spaces similar to those that were obtained in [17] for the complex case. Namely it is proved that a bounded group of J-unitary operators in a real Pontryagin space has a maximal negative invariant subspace and that a bounded representation of a group in a real Hilbert space that preserves a sesquilinear form with a finite number of negative squares is similar to an orthogonal representation. We also remove the assumption of separability of H made in [17].

Our main technical tool is the proof of a kind of normal structure in $\mathcal{B}$ - the existence of non-diametral points for a sufficiently rich class of sets. In more detail: first we establish that each pair $X, Y \in \mathcal{B}$ has at least one *midpoint*, that is, a point $Z \in \mathcal{B}$ satisfying $\rho(Z, W) \leq 1/2(\rho(X, W) + \rho(Y, W))$ for all $W \in \mathcal{B}$. We call a subset of $\mathcal{B}$ *mid-convex* if with any two points it contains all of their midpoints. WOT-compact mid-convex subsets of $\mathcal{B}$ are called *m-sets*. Then we prove that each m-set either is a singleton or has a non-diametral point (Theorem 3.8).

The next step is to show that the presence of a ρ-bounded orbit implies the existence of an invariant m-set. After that we use WOT-compactness and get the existence of a minimal invariant m-set. Then we apply Theorem 3.8 to prove that such minimal set should be a singleton.

The described general scheme for proving fixed point theorems for nonexpansive mappings goes back to [3]. This scheme was gradually developed by many different authors, and a large variety of realizations of it was found in further work on fixed points of nonexpansive mappings, see [6], [11], [12], [20], [21], numerous examples of applications of this scheme can be found in [13].

Note that another instance of the connection between boundedness of orbits and the existence of fixed points can be found in [20].

2   Metric geometry: the midpoint property, fixed points of groups isometries, and other tools

Let $(\mathcal{X}, d)$ be a metric space. By a *ball* in $(\mathcal{X}, d)$ we mean (unless it is explicitly stated otherwise) a closed ball $E_{a,r} = \{x \in X : d(a, x) \leq r\}$. We say that $(\mathcal{X}, d)$ is *ball-compact* if a family of balls has non-void intersection provided each its finite subfamily has non-void intersection (see [10]). It is easy to show that each ball-compact metric space is complete.

A subset $M \subset \mathcal{X}$ is called *ball-convex* if it is an intersection of a family of balls. It is clear from the definition that each ball-convex set is bounded and closed. The compactness extends from balls to all ball-convex sets:

Lemma 2.1 ([10])**.** *Let $(\mathcal{X}, d)$ be ball-compact. A family $\{M_\lambda : \lambda \in \Lambda\}$ of ball-convex subsets of $\mathcal{X}$ has non-empty intersection if each its finite subfamily has non-empty intersection.*

Proof. For each $\lambda \in \Lambda$, we have $M_\lambda = \cap_{i \in I_\lambda} E_i^\lambda$ where all E_i^λ are balls. Let us consider the family of balls $\mathcal{U} = \{E_i^\lambda : \lambda \in \Lambda, i \in I_\lambda\}$. Each finite subfamily of $\mathcal{U}$ has a non-void intersection because it contains the intersection of corresponding M_λ. By ball-compactness, $\mathcal{U}$ has non-zero intersection which clearly coincides with $\cap_\lambda M_\lambda$. □

The *diameter* of a subset $M \subset \mathcal{X}$ is defined by

$$\operatorname{diam} M = \sup\{d(x, y) : x, y \in M\}. \tag{2.1}$$

A point a in a bounded subset M is called *diametral* if

$$\sup\{d(a, x) : x \in M\} = \operatorname{diam} M.$$

A metric space $\mathcal{X}$ is said to have *normal structure* if every ball-convex subset of $\mathcal{X}$ with more than one element has a non-diametral point. A mapping $g : \mathcal{X} \to \mathcal{X}$ is called *nonexpansive* if $d(g(x), g(y)) \le d(x, y)$ for all $x, y \in \mathcal{X}$.

The concept of normal structure, introduced by Brodskii and Milman [3] for Banach spaces, has played a prominent role in fixed point theory for nonexpansive mappings. See, for example, [6], [11], [13], [21], [22]. In particular the scheme of the proof of the following result is well known; nevertheless the result itself seems to be new.

Theorem 2.2. *Suppose that a metric space $(\mathcal{X}, d)$ is ball-compact and has normal structure. If a group of isometries of $(\mathcal{X}, d)$ has a bounded orbit, then it has a fixed point.*

Proof. Let G be a group of isometries of $(\mathcal{X}, d)$ and let $G(x)$ be a bounded orbit, where x is some point in $\mathcal{X}$. Then the family Φ of all balls containing $G(x)$ is non-empty. Since $G(x)$ is invariant under G, the family Φ is also invariant: $g(E) \in \Phi$, for each $E \in \Phi$. Hence the intersection M_1 of all elements of Φ is a non-void G-invariant ball-convex set.

Thus the family $\mathcal{M}$ of all non-void G-invariant ball-convex subsets of $\mathcal{X}$ is non-empty. It follows from Lemma 2.1 that the intersection of a decreasing chain of sets in $\mathcal{M}$ belongs to $\mathcal{M}$. By Zorn Lemma, $\mathcal{M}$ has minimal elements. Our aim is to prove that a minimal element M of $\mathcal{M}$ consists of one point.

Assume the contrary and let $\operatorname{diam} M = \alpha > 0$. Since $(\mathcal{X}, d)$ has normal structure, M contains a non-diametral point a. It follows that $M \subset \{x \in \mathcal{X} : d(a, x) \le \delta\}$ for some $\delta < \alpha$. Set

$$O = \bigcap_{b \in M} E_{b, \delta}.$$

The set O is non-empty because $a \in O$. Furthermore O is ball-convex by definition. To see that O is a proper subset of M take $b, c \in M$ with $d(b, c) > \delta$, then $c \notin E_{b, \delta}$, hence $c \notin O$.

Since G is a group of isometric transformations and M is invariant under each element of G, the action of G on M is by isometric bijections. Therefore O is G-invariant. We get a contradiction with the minimality of M. $\qquad\square$

The following definition is essential for our results on normal structure and non-diametral points.

Definition 2.3. A metric space $(\mathcal{X}, d)$ is said to have *the midpoint property* if for any two points $a, b \in \mathcal{X}$ there is $c \in \mathcal{X}$ such that

$$d(c, x) \le (d(a, x) + d(b, x))/2 \quad \forall \, x \in (\mathcal{X}, d).$$

Each point c satisfying this condition is called a *midpoint* for (a, b), the set of all midpoints for (a, b) is denoted by $m(a, b)$.

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Remark 2.4. A straightforward adaptation of the well-known argument of Menger [16] (see also [2, Theorem 14.1]) shows that complete metric spaces with the midpoint property have the *convex structure* defined by Takahashi [25], but we do not need this fact.

Our proof of the fact that operator balls have the midpoint property uses the following definition and lemma. We say that an isometric involution σ of $\mathcal{X}$ is *a reflection in a point* $x_0 \in \mathcal{X}$ if $d(x, \sigma(x)) = 2d(x, x_0)$ for each $x \in \mathcal{X}$.

Lemma 2.5. *If there exists a reflection σ in a point $x_0 \in \mathcal{X}$, then x_0 is a midpoint for each pair $(x, \sigma(x))$.*

Proof. Let $x \in \mathcal{X}$. For each $y \in \mathcal{X}$, $d(y, x_0) = \frac{1}{2}d(y, \sigma(y)) \leq \frac{1}{2}(d(y, x) + d(\sigma(y), x))) = \frac{1}{2}(d(y, x) + d(y, \sigma(x)))$. This is what we need. $\qquad\square$

Our next result is the main technical tool for proving results on normal structure for spaces with the midpoint property.

Lemma 2.6. *Suppose that a metric space $(\mathcal{X}, d)$ has the midpoint property. Let M be a ball-convex subset of $\mathcal{X}$ and α be the diameter of M. If all points of M are diametral, then M contains a net $\{c_\lambda : \lambda \in \Lambda\}$ with the property: $\lim_\lambda d(c_\lambda, x) = \alpha$ for each $x \in M$.*

Proof. For $a_1, ..., a_n \in \mathcal{X}$, we denote by $m(a_1, ..., a_n)$ the set of all points $a \in \mathcal{X}$ satisfying $d(a, x) \leq (d(a_1, x) + ... + d(a_n, x))/n$ for each $x \in \mathcal{X}$. The easy induction argument shows that if $\mathcal{X}$ has the midpoint property and $n = 2^k$, then $m(a_1, ..., a_n)$ is nonempty.

Let $F = (a_1, ..., a_n)$ be a finite subset of M and let $\varepsilon > 0$. We will show that there is a point $c = c(F, \varepsilon)$ such that $d(a_i, c) \geq (1 - \varepsilon)\alpha$ for all i. It is easy to see that a set with the midpoint property is either a singleton or it is infinite. Adding new points we may assume that $n = 2^k$. Let $b \in m(a_1, ..., a_n)$. If $F \subset E_{a,r}$ then $d(b, a) \leq (d(a_1, a) + ... + d(a_n, a))/n \leq r$ so $b \in E_{a,r}$. Thus b belongs to each ball that contains M; by ball-convexity, $b \in M$.

Since b is diametral, there is $c \in M$ with $d(b, c) \geq (1 - \frac{\varepsilon}{n})\alpha$. It follows that $(1 - \frac{\varepsilon}{n})\alpha \leq \frac{1}{n}\sum_{k=1}^{n} d(a_k, c)$. If $d(a_j, c) < (1 - \varepsilon)\alpha$, for some $j \leq n$, then

$$\frac{1}{n}\sum_{k=1}^{n} d(a_j, c) < \frac{1}{n}(1 - \varepsilon)\alpha + \frac{n-1}{n}\alpha = (1 - \frac{\varepsilon}{n})\alpha,$$

a contradiction. Hence $d(a_j, c) \geq (1 - \varepsilon)\alpha$ for $j \leq n$.

Let now Λ be the set of all pairs (F, ε) with the order given by: $(F_1, \varepsilon_1) \prec (F_2, \varepsilon_2)$ if $F_1 \subset F_2, \varepsilon_1 > \varepsilon_2$. For $\lambda = (F, \varepsilon) \in \Lambda$, we write $c_\lambda = c(F, \varepsilon)$. It is clear that $\lim_\lambda d(c_\lambda, x) = \alpha$ for each $x \in M$. $\qquad\square$

The class of ball-convex sets is not sufficiently rich for our needs, because it is not closed with respect to unions of increasing families of sets. For this reason we introduce the following class of sets, which does not have this drawback.

Definition 2.7. A subset M of a metric space is called *mid-convex* if $m(x, y) \subset M$ for each pair (x, y) of points in M.

Clearly each ball is mid-convex. Since the class of all mid-convex sets is closed under intersection, each ball-convex set is mid-convex. It is also clear that each mid-convex subset M of a metric space with the midpoint property is itself a metric space with the midpoint property. This shows that Lemma 2.6 admits the following extension:

Lemma 2.8. *Let M be a bounded mid-convex subset of a metric space $(\mathcal{X}, d)$ having the midpoint property. If all points of M are diametral, then M contains a net $\{c_\lambda : \lambda \in \Lambda\}$ with the property:* $\lim_\lambda d(c_\lambda, x) = \operatorname{diam} M$ *for each $x \in M$.*

3 Fixed points for groups of biholomorphic maps on the operator ball

Let K, H be real or complex Hilbert spaces; by $L(K, H)$ we denote the Banach space of all bounded linear operators from K to H. We denote the open unit ball of $L(K, H)$ by $\mathcal{B}$ and call it *operator ball*. The main goal of this section is to show that if K and H are complex spaces and $\dim K < \infty$, then the operator ball possesses a metric ρ which is invariant with respect to biholomorphic transformations of $\mathcal{B}$ and satisfies the conditions of Theorem 2.2, and therefore any group of biholomorphic automorphisms of the operator ball with a bounded orbit has a fixed point.

3.1 Möbius transformations and invariant distance in the operator ball

For each $A \in \mathcal{B}$, we define a transformation M_A of $\mathcal{B}$ (*a Möbius transformation*) setting

$$M_A(X) = (1 - AA^*)^{-1/2}(A + X)(1 + A^*X)^{-1}(1 - A^*A)^{1/2} \tag{3.1}$$

(this definition is due to Potapov [19]). It can be proved that

$$M_A^{-1} = M_{-A} \tag{3.2}$$

(see [7], Theorem 2).

We set

$$\rho(A, B) = \tanh^{-1}(||M_{-A}(B)||). \tag{3.3}$$

It is easy to see that if K, H are complex, then ρ coincides with the Carathéodory distance $c_\mathcal{B}$ in $\mathcal{B}$. Indeed, by [5, Theorem 4.1.8], $c_\mathcal{B}(0, B) = \tanh^{-1}(||B||)$ (this holds for the unit ball of every Banach space). Furthermore all M_A are clearly holomorphic mappings of $\mathcal{B}$, and the formula (3.2) shows that the same is true for their inverses. In other words, M_A are biholomorphic automorphisms of $\mathcal{B}$. Since $c_\mathcal{B}$ is invariant with respect to biholomorphic automorphisms and M_{-A} sends A to 0, we get:

$$c_\mathcal{B}(A, B) = \tanh^{-1}||M_{-A}(B)|| = \rho(A, B). \tag{3.4}$$

The equality (3.4) shows that ρ is a metric on $\mathcal{B}$.

To check that the same is true in the case of real scalars, one can use the complexification. Indeed, setting $\tilde{H} = H \oplus iH$, $\tilde{K} = K \oplus iK$ one defines maps $A \to \tilde{A}$ from $L(H)$ to $L(\tilde{H})$ (and from $L(K)$ to $L(\tilde{K})$, from $L(K, H)$ to $L(\tilde{K}, \tilde{H})$, from $L(H, K)$ to $L(\tilde{H}, \tilde{K})$) by $\tilde{A}(x \oplus iy) = Ax \oplus iAy$. These maps are isometric *-homomorphisms of algebras and modules whence $\widetilde{M_A(B)} = M_{\tilde{A}}(\tilde{B})$ and $\rho(A, B) = \rho(\tilde{A}, \tilde{B})$.

3.2   Midpoint property of $(\mathcal{B}, \rho)$

Our next aim is to prove that the metric space $(\mathcal{B}, \rho)$ has the midpoint property.

Lemma 3.1. $\rho(A, -A) = 2\rho(0, A)$.

Proof. Let $A = UT$ be the polar decomposition of A (so $T = |A| = (A^*A)^{1/2}$). Then

$$
\begin{aligned}
\rho(A, -A) &= \tanh^{-1} \|M_A(A)\| = \tanh^{-1} \|(1 - AA^*)^{-1/2} 2A(1 + A^*A)^{-1}(1 - A^*A)^{1/2}\| \\
&= \tanh^{-1} \|2A(1 - A^*A)^{-1/2}(1 + A^*A)^{-1}(1 - A^*A)^{1/2}\| \\
&= \tanh^{-1} \|U2T(1 + T^2)^{-1}\| = \tanh^{-1} \|2T(1 + T^2)^{-1}\| = \|\tanh^{-1}(2T(1 + T^2)^{-1})\| \\
&= \|2\tanh^{-1}(T)\| = 2\tanh^{-1}\|T\| = 2\tanh^{-1}\|A\| = 2\rho(0, A)
\end{aligned}
$$

(we used the identity $\tanh(2x) = 2\tanh x/(1 + \tanh^2 x)$ and the fact that $\|f(T)\| = f(\|T\|)$ if a function f is non-decreasing on the interval $[0, \|T\|]$ and T is a positive operator). $\square$

Lemma 3.2. *For every $A, B \in \mathcal{B}$, there is an isometric transformation φ of $(\mathcal{B}, \rho)$ with $\varphi(A) = -\varphi(B)$.*

Proof. It suffices to show this for $A = 0$, because the Möbius transformations (which are isometries of $(\mathcal{B}, \rho)$) act transitively on $\mathcal{B}$. Thus we have to find C, D in $\mathcal{B}$ such that $M_C(-D) = 0$, $M_C(D) = B$. Take $D = C$, then we have the equation $M_C(C) = B$. By the above calculations, this means that $U2T(1 + T^2)^{-1} = B$, where $UT = C$ is the polar decomposition. If $B = VS$ is the polar decomposition of B, then we take $U = V$ and $T = \tanh(\frac{1}{2}\tanh^{-1}(S))$. $\square$

Proposition 3.3. *The space $(\mathcal{B}, \rho)$ has the midpoint property.*

Proof. Let $A, B \in \mathcal{B}$; we have to prove that there is $C \in \mathcal{B}$ such that $\rho(X, C) \leq \frac{1}{2}(\rho(X, A) + \rho(X, B))$ for each $X \in \mathcal{B}$. By Lemma 3.2, we may assume that $B = -A$. The involution $\sigma(X) = -X$ is clearly isometric; it follows from Lemma 3.1 that it is a reflection in the point 0. Now by Lemma 2.5, 0 is a midpoint for the pair $(A, -A)$. $\square$

3.3 Topologies on $\mathcal{B}$ and ball-compactness

A set in $\mathcal{B}$ is bounded with respect to ρ if it is contained in some ρ-ball, this happens if and only if the set is contained in a ρ-ball with center 0. On the other hand, it follows from the definition of ρ that a ρ-ball with center 0 coincides with the closure of a multiple $r\mathcal{B}$ of the operator ball $\mathcal{B}$, for some $r < 1$. Therefore a set is bounded if and only if it is separated from the boundary of $\mathcal{B}$.

The following lemma is a special case of a more general result proved in [5, Theorem IV.2.2] (in the case of real scalars one can use the complexification).

Lemma 3.4. *The metric ρ is equivalent to the operator norm on any ρ-bounded set.*

Thus the topology induced by the metric ρ on $\mathcal{B}$ is equivalent to the norm topology.

Another important topology on $\mathcal{B}$ is the weak operator topology WOT (see [4, p. 476]). We collect the facts about the WOT that we need in the following lemma.

Lemma 3.5. (i) *Each ρ-ball $E_{A,r}$ is convex and WOT-compact.*
(ii) *If $\dim K < \infty$, then each norm-closed convex subset of $\mathcal{B}$ is WOT-compact.*
(iii) *If $\dim K < \infty$, then all Möbius transformations are WOT-continuous.*

Proof. (i) By definition, each ρ-ball is the image of a ρ-ball with center 0 under a Möbius transform. Since 0-centered ρ-balls are the usual balls and since a Möbius transform is a fractional-linear transformation, the statement follows from the fact that images of balls under fractional-linear transformations are convex and WOT-compact (see, for example, [9]).

(ii) It is easy to check that if K is finite-dimensional, then the norm topology of $L(K, H)$ coincides with the strong operator topology while WOT coincides with the weak topology of $L(K, H)$. Since convex norm-closed bounded subsets of a reflexive space are weakly compact, the result follows.

(iii) The WOT-continuity of a more general class of maps ("fractional-linear maps with compact (1,2)-entries") was noticed and used by Krein in [15]. □

Corollary 3.6. (i) *The metric space $(\mathcal{B}, \rho)$ is ball-compact.*
(ii) *Ball-convex subsets of $(\mathcal{B}, \rho)$ are WOT-compact.*

3.4 The normal structure of $(\mathcal{B}, \rho)$

Recall that the normal structure means the existence of non-diametral points in ball-convex sets. We establish the existence of non-diametral points for a wider class of subsets of $\mathcal{B}$.

Definition 3.7. A subset M of $\mathcal{B}$ is called an *m-set* if it is mid-convex and WOT-compact.

It follows from the results of the previous section that the class of all m-sets contains the class of all ball-convex sets. Furthermore, if M is a ρ-bounded (= separated from the boundary) subset of $\mathcal{B}$, then there is a smallest m-set M_0 containing M (the intersection of all m-sets containing M). We call M_0 *the m-span of M*.

Theorem 3.8. *Let K be finite dimensional. Then any m-set in $\mathcal{B}$ either is a singleton or has a non-diametral point.*

Proof. Let M be an m-set of $(\mathcal{B}, \rho)$ which is not a singleton. We have to prove that M contains a non-diametral point.

Assume the contrary, that is, all points in M are diametral. Let $\alpha = \operatorname{diam} M > 0$. Since $\mathcal{B}$ and therefore M has the midpoint property, we have, by Lemma 2.8, that there is a net $\{A_\lambda\}$ in M such that $\lim_\lambda \rho(A_\lambda, X) = \alpha$ for each $X \in M$.

Since M is WOT-compact, the net $\{A_\lambda : \lambda \in \Lambda\}$ contains a weakly convergent cofinal subnet. To simplify the notation we assume that the net $\{A_\lambda : \lambda \in \Lambda\}$ itself WOT-converges to some operator W. Since M is WOT-compact, we have $W \in M$.

Since $W \in M$, we get

$$\lim_\lambda \rho(W, A_\lambda) = \alpha. \tag{3.5}$$

We get a contradiction by proving

$$\sup_{\lambda, \mu} \rho(A_\lambda, A_\mu) > \alpha. \tag{3.6}$$

Since all Möbius transformations are WOT-continuous isometries of $(\mathcal{B}, \rho)$ (see Lemma 3.5(iii)), we may assume without loss of generality that $W = 0$ (otherwise we apply M_{-W}).

Let $\beta = \tanh \alpha$. Then (3.5) leads to $\lim_\lambda \|A_\lambda\| = \beta$ and it suffices to show that

$$\sup_{\lambda, \mu} \|M_{A_\mu}(-A_\lambda)\| > \beta.$$

Since K is finite dimensional and $A_\lambda \in L(K, H)$, we can select a strongly convergent subnet in the net $\{A_\lambda^* A_\lambda\}$. So we assume that $A_\lambda^* A_\lambda \to P$, where $P \in L(K, K)$. It is clear that $P \geq 0$ and $\|P\| = \beta^2$.

Choose $\varepsilon > 0$ and fix μ with $\|A_\mu^* A_\mu - P\| < \varepsilon$. For brevity, denote $A_\mu^* A_\mu$ by Q. We prove that $\lim_\lambda \|M_{A_\mu}(-A_\lambda)\| > \beta$ if $\varepsilon > 0$ is small enough. By the definition,

$$M_{A_\mu}(-A_\lambda) = (1 - A_\mu A_\mu^*)^{-1/2}(A_\mu - A_\lambda)(1 - A_\mu^* A_\lambda)^{-1}(1 - A_\mu^* A_\mu)^{1/2}. \tag{3.7}$$

Since A_μ^* is of finite rank, $A_\mu^* A_\lambda \to 0$ in the norm topology. Hence $\lim_\lambda \|M_{A_\mu}(-A_\lambda)\| = \lim_\lambda \|T_\lambda\|$, where

$$T_\lambda = (1 - A_\mu A_\mu^*)^{-1/2}(A_\mu - A_\lambda)(1 - A_\mu^* A_\mu)^{1/2} = A_\mu - (1 - A_\mu A_\mu^*)^{-1/2} A_\lambda (1 - A_\mu^* A_\mu)^{1/2}.$$

It follows from the identity

$$(1 - t)^{-1/2} - 1 = \frac{t}{(1 - t)(1 + (1 - t)^{-1/2})}$$

that the operator $(1 - A_\mu A_\mu^*)^{-1/2}$ is a finite rank perturbation of the identity operator. Since $A_\lambda \to 0$ in WOT, we obtain that $\|T_\lambda - S_\lambda\| \to 0$, where $S_\lambda = A_\mu - A_\lambda (1 - A_\mu^* A_\mu)^{1/2}$.

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Denote $A_\lambda(1 - A_\mu^* A_\mu)^{1/2}$ by B_λ. Since $B_\lambda \to 0$ in WOT, the sequence

$$(A_\mu - B_\lambda)^*(A_\mu - B_\lambda) - A_\mu^* A_\mu - B_\lambda^* B_\lambda = -A_\mu^* B_\lambda - B_\lambda^* A_\mu$$

tends to zero in norm topology. Furthermore,

$$B_\lambda^* B_\lambda = (1 - Q)^{1/2} A_\lambda^* A_\lambda (1 - Q)^{1/2}$$

tends in norm topology to $(1 - Q)^{1/2} P (1 - Q)^{1/2}$. Therefore

$$(A_\mu - B_\lambda)^*(A_\mu - B_\lambda) \to Q + (1 - Q)^{1/2} P (1 - Q)^{1/2}.$$

Since $\|P - Q\| < \varepsilon$, we have that

$$\|Q + (1 - Q)^{1/2} P (1 - Q)^{1/2} - (Q + (1 - Q)Q)\| < \varepsilon.$$

The inequalities

$$\beta^2 - \varepsilon \le \|Q\| \le \beta^2$$

imply

$$\|Q + (1 - Q)Q\| \ge 2\beta^2 - \beta^4 - 2\varepsilon,$$

whence

$$\lim_\lambda \|S_\lambda^* S_\lambda\| = \lim_\lambda \|(A_\mu - B_\lambda)^*(A_\mu - B_\lambda)\| \ge 2\beta^2 - \beta^4 - 3\varepsilon > \beta^2,$$

the last inequality is satisfied if ε is sufficiently small. □

Since $(\mathcal{B}, \rho)$ has the midpoint property (Proposition 3.3), all ball-convex sets in $\mathcal{B}$ are mid-convex and we obtain

Corollary 3.9. *The metric space $(\mathcal{B}, \rho)$ has normal structure.*

By Corollaries 3.6, 3.9, and Theorem 2.2, we immediately get

Theorem 3.10. *Let $\dim K < \infty$. If a group G of ρ-isometric maps of $\mathcal{B}$ has at least one orbit separated from the boundary, then it has a fixed point.*

Since biholomorphic mappings of $\mathcal{B}$ are isometric with respect to the Carathéodory metric, we get also

Corollary 3.11. *Let H and K be complex, $\dim K < \infty$. If a group G of biholomorphic automorphisms of $\mathcal{B}$ has at least one orbit separated from the boundary, then it has a fixed point.*

4 Fixed points for commuting holomorphic maps on the operator ball

We continue to suppose that $\dim K < \infty$. The spaces can be real or complex.

Lemma 4.1. *Let G be a commutative semigroup of ρ-nonexpansive maps of $\mathcal{B}$. If it has a ρ-bounded orbit, then it has an invariant m-set.*

Proof. Let $X \in \mathcal{B}$ be an operator whose orbit $G(X)$ is ρ-bounded. We introduce the following partial order on G: $g \succ h$ if and only if $g = hf$ for some $f \in G$. The semigroup G is a directed set with respect to this order (we use the standard definitions from [4, Chapter I]).

Let $r = \mathrm{diam}(G(X))$ and let $F = \{Y \in \mathcal{B} : \limsup_{g \in (G, \succ)} \rho(Y, g(X)) \leq r\}$. Then

$$F = \bigcap_{\varepsilon > 0} \bigcup_{g \in G} \bigcap_{h \succ g} E_{h(X), r+\varepsilon}.$$

Since each ρ-ball $E_{Z,p}$ is convex in the usual sense (see Lemma 3.5(i)) and is mid-convex, the same is true for F. Also it is easy to see that F is ρ-bounded and is closed in the topology defined by the metric ρ. By Lemma 3.4, F is norm-closed. By Lemma 3.5(ii), F is WOT-compact, thus F is an m-set.

Let us check that F is G-invariant. Indeed, if $Y \in F$ and $h \in G$, then

$$\limsup_{g \in (G, \succ)} \rho(h(Y), g(X)) = \limsup_{f \in (G, \succ)} \rho(h(Y), hf(X)) \leq \limsup_{f \in (G, \succ)} \rho(Y, f(X)) \leq r.$$

This means that $h(Y) \in F$. $\square$

The idea of our proof of Lemma 4.1 goes back to [11].

Lemma 4.2. *If a ρ-nonexpansive map $g : \mathcal{B} \to \mathcal{B}$ has an invariant bounded set U, then it has a fixed point in $\mathcal{B}$.*

Moreover, if E is an arbitrary invariant m-set containing U, then a fixed point can be found in E.

Proof. The assumption is equivalent to the condition that the semigroup generated by g has a bounded orbit. By Lemma 4.1, there is an invariant m-set for g. Since m-sets are WOT-compact, there exists a minimal invariant m-set F for g. The m-span F_1 of $g(F)$ is contained in F and is invariant (indeed $g(F_1) \subset g(F) \subset F_1$). Hence $F_1 = F$. If F is a singleton, then it is a fixed point.

Now assume that F is not a singleton. By Theorem 3.8, there is $r < \mathrm{diam}(F)$ such that the set

$$F(r) := \{Y \in F : \forall Z \in F \ \rho(Y, Z) \leq r\}$$

is non-empty. Since $F(r)$ is the intersection of F with a family of balls, it is an m-set. Moreover $F(r)$ is invariant. Indeed, if $Y \in F(r)$, then $\rho(g(Y), g(X)) \leq r$ for all $X \in F$. Hence the ball $E_{g(Y), r}$ contains $g(F)$. Since F is the m-span of $g(F)$, this ball contains F.

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Hence $g(Y) \in F(r)$. It is also clear that $F(r)$ is a proper subset of F. We get a contradiction with the minimality of F.

The second statement of the lemma follows from the construction. $\qquad\square$

Using Lemma 4.2 we prove the following analogue of Shafrir's theorem [24, Theorem 3.2] for $\mathcal{B}$:

Theorem 4.3. *A finite set $\mathcal{E}$ of commuting ρ-nonexpansive maps of $\mathcal{B}$ has a fixed point in $\mathcal{B}$ if and only if $\mathcal{E}$ has an invariant set separated from the boundary of $\mathcal{B}$.*

Proof. The "only if" part is trivial, we prove the "if" part only. Recall that a set is separated from the boundary if and only if it is ρ-bounded. If a set $\mathcal{E} = \{g_1, ..., g_n\}$ has a ρ-bounded invariant set, then the semigroup generated by $\mathcal{E}$ has a bounded orbit. By Lemma 4.1, there is an invariant m-set for $\mathcal{E}$. Let F be a minimal invariant m-set for $\mathcal{E}$.

Let $g = g_1 g_2 ... g_n$. By Lemma 4.2, g has a fixed point in F. Let R be the set of all fixed points of g in F.

We claim that $g_i(R) = R$ for each i. The inclusion $g_i(R) \subset R$ follows from commutativity. On the other hand, if $X \in R$, then $X = g(X) = g_i(Y)$, where $Y = h(X)$ and h is the product of all g_j with $j \neq i$. Clearly $Y \in R$, $X \in g_i(R)$, thus, $R \subset g_i(R)$.

Let K be the smallest m-set containing R, clearly $K \subset F$. If K is a singleton, then $R = K$ contains only one point which is a common fixed point for $g_1, \ldots, g_n$.

If K is not a singleton, then, by Theorem 3.8, there is $r < \text{diam}(K)$ such that the set $K_r = \{X \in K : \rho(X, Y) \leq r, \text{ for all } Y \in K\}$ is non-empty. Since K_r is an intersection of an m-set with a family of ρ-balls, it is also an m-set.

Let us show that K_r is invariant under all g_i, $i = 1, \ldots, n$. First we note that $K_r = \{X \in K : \rho(X, Y) \leq r, \text{ for all } Y \in R\}$. Indeed, if $\rho(X, Y) \leq r$ for all $Y \in R$, then $R \subset E_{X,r}$ whence $K \subset E_{X,r}$ because $E_{X,r}$ is an m-set. Now if $X \in K_r$, $Y \in R$, using the fact that $g_i(R) = R$, we obtain $\rho(g_i(X), Y) = \rho(g_i(X), g_i(Y')) \leq \rho(X, Y') \leq r$. By minimality of F, we get $K_r = F$. We get a contradiction since $K \subset F$ and K_r is a proper subset of K. $\qquad\square$

Note that Theorem 4.3 extends to infinite families of ρ-nonexpansive maps if these maps are assumed to be WOT-continuous (indeed in this case the set of all fixed points for each finite family of maps is WOT-compact).

Since holomorphic maps are ρ-nonexpansive, Theorem 4.3 has the following corollary.

Corollary 4.4. *Let K and H be complex spaces. A finite set $\mathcal{E}$ of commuting holomorphic maps of $\mathcal{B}$ has a common fixed point in $\mathcal{B}$ if and only if $\mathcal{E}$ has an invariant set separated from the boundary of $\mathcal{B}$.*

5   Operator-theoretic applications: the case of real spaces

For applications to invariant subspaces and orthogonalization, we need to prove that ρ is invariant under maps on $\mathcal{B}$ which are induced by J-unitary operators of the space $\mathcal{H} = H \oplus K$. In the case of complex scalars this follows immediately from the fact that these maps are holomorphic, while in the real case we need some additional arguments.

We remind some definitions and constructions from the theory of spaces with an indefinite product. We refer to [1] or [14] for more information.

An operator U on $\mathcal{H}$ is called J-*unitary* if it preserves the indefinite scalar product

$$[x, y] = (P_H x, y) - (P_K x, y), \tag{5.1}$$

where P_H and P_K are the projections on the summands in the decomposition $\mathcal{H} = H \oplus K$. To each J-unitary operator U on $\mathcal{H}$ there corresponds a map w_U of $\mathcal{B}$ in such a way that

$$w_{U_1 U_2} = w_{U_1} \circ w_{U_2}. \tag{5.2}$$

To see this recall that a vector $x \in \mathcal{H}$ is called *positive* (*neutral, negative*) if $[x, x] > 0$ ($[x, x] = 0$, $[x, x] < 0$, respectively). A subspace of $\mathcal{H}$ is called *positive* (*neutral, negative*) if all its non-zero elements are positive (neutral, negative, respectively). For each operator $X \in \mathcal{B}$ the set

$$S(X) = \{Xx \oplus x : x \in K\}$$

is a negative subspace of $\mathcal{H}$. Since $\dim(S(X)) = \dim(K)$, $S(X)$ is a maximal negative subspace in $\mathcal{H}$. Indeed, if some subspace M of $\mathcal{H}$ strictly contains $S(X)$, then its dimension is greater than the codimension of H, whence $M \cap H \neq \{0\}$. But all non-zero vectors in H are positive.

Conversely, each maximal negative subspace Q of $\mathcal{H}$ coincides with $S(X)$, for some $X \in \mathcal{B}$. Indeed, since $Q \cap H = \{0\}$, there is an operator $X : K \to H$ such that each vector of Q is of the form $Xx \oplus x$. Since Q is negative, we have $[Xx \oplus x, Xx \oplus x] = \|Xx\|^2 - \|x\|^2 < 0$. Since K is finite dimensional, this implies $\|X\| < 1$, so $X \in \mathcal{B}$. Thus $Q \subset S(X)$; and, by maximality, $Q = S(X)$.

It is easy to see that the map $X \to S(X)$ from $\mathcal{B}$ to the set $\mathcal{E}$ of all maximal negative subspaces is injective and therefore bijective.

Now we can define w_U. Note that if a subspace L of $\mathcal{H}$ is maximal negative, then its image UL under U is also maximal negative (because U is invertible and preserves $[\cdot, \cdot]$). Hence, for each $X \in \mathcal{B}$, there is $Y \in \mathcal{B}$ such that $S(Y) = US(X)$. We let $w_U(X) = Y$.

The equality (5.2) follows easily because

$$S(w_{U_1}(w_{U_2}(X))) = U_1 S(w_{U_2}(X)) = U_1 U_2 S(X) = S(w_{U_1 U_2}(X))$$

and the map $X \to S(X)$ is injective.

Let $U = (U_{ij})_{i,j=1}^2$ be the matrix of U with respect to the decomposition $\mathcal{H} = H \oplus K$. Then $U(Xx \oplus x) = (U_{11}Xx + U_{12}x) \oplus (U_{21}Xx + U_{22}x)$. Since $U(Xx \oplus x) \in S(w_U(X))$, we conclude that

$$w_U(X)(U_{21}Xx + U_{22}x) = U_{11}Xx + U_{12}x.$$

Thus

$$w_U(X) = (U_{11}X + U_{12})(U_{21}X + U_{22})^{-1}. \tag{5.3}$$

Let us denote by $\mathcal{T}$ the group of all such transformations of $\mathcal{B}$ (Helton [8] calls them *general symplectic transformations*). It should be noted that all Möbius maps belong to $\mathcal{T}$. Namely $M_A = w_U$ where U is the J-unitary operator with the matrix $(U_{ij})_{i,j=1}^2$, where

$$U_{11} = (1_H - AA^*)^{-1/2}, U_{12} = -A(1_K - A^*A)^{-1/2},$$
$$U_{21} = -A^*(1_H - AA^*)^{-1/2}, U_{22} = (1_K - A^*A)^{-1/2}.$$

If a map $\varphi \in \mathcal{T}$ has the property $\varphi(0) = 0$, then it can be written in the form $\varphi(X) = V_1XV_2$ where V_1, V_2 are unitary operators in H and K, respectively. Indeed, let U be a J-unitary operator with $\varphi = w_U$, then $UK = US(0) = S(\varphi(0)) = S(0) = K$. Since H is a $[\cdot, \cdot]$-orthogonal complement of K in $\mathcal{H}$, it is also invariant under U. Thus U has a diagonal matrix with respect to the decomposition $\mathcal{H} = H \oplus K$: $U_{12} = 0$, $U_{21} = 0$. A moment reflection shows that U_{11} and U_{22} are unitary operators in H and K, respectively. Thus $\varphi(X) = U_{11}XU_{22}^{-1}$ is of the needed form.

It follows now that $\|\varphi(X)\| = \|X\|$ for all $X \in \mathcal{B}$, if $\varphi(0) = 0$.

If $\varphi \in \mathcal{T}$ sends A to 0, then the transformation $\varphi_1 = \varphi \circ M_A$ preserves 0 and sends $M_{-A}(B)$ to $\varphi(B)$. Thus, by the previous statement, $\|\varphi(B)\| = \|M_{-A}(B)\|$ and $\rho(A, B) = \tanh^{-1}\|\varphi(B)\| = \rho(0, \varphi(B))$ for each $\varphi \in \mathcal{T}$ with $\varphi(A) = 0$.

This implies that $\rho(\varphi(A), \varphi(B)) = \rho(A, B)$ for all $A, B \in \mathcal{B}$ and $\varphi \in \mathcal{T}$. Indeed, let $\alpha \in \mathcal{T}$ with $\alpha(A) = 0$, then $\rho(A, B) = \rho(0, \alpha(B))$. Setting $\psi = \alpha \circ \varphi^{-1}$ we get that $\psi(\varphi(A)) = 0$ so

$$\rho(\varphi(A), \varphi(B)) = \rho(0, \psi(\varphi(B))) = \rho(0, \alpha(B)) = \rho(A, B).$$

Thus $\mathcal{T}$ is a group of isometries of $(\mathcal{B}, \rho)$. Applying Theorem 3.10 we obtain

Corollary 5.1. *If a group G of general symplectic transformations of $\mathcal{B}$ has an orbit separated from the boundary of $\mathcal{B}$, then it has a fixed point in $\mathcal{B}$.*

Recall that subspaces M, N of a space with indefinite scalar product $\mathcal{H}$ form *a dual pair of subspaces* if M is positive, N is negative and $M + N = \mathcal{H}$.

A *Pontryagin space* Π_k is the space $\mathcal{H} = H \oplus K$ with the scalar product (5.1), where H and K are Hilbert spaces, $\dim H = \infty$, $\dim K = k$.

Corollary 5.2. *Any bounded group of J-unitary operators in a real Pontryagin space Π_k has an invariant dual pair of subspaces.*

Proof. Let Γ be a bounded group of J-unitary operators in $\mathcal{H}$, $C = \sup_{U \in \Gamma} \|U\|$ and $G = \{w_U : U \in \Gamma\}$. Then the orbit $G(0)$ is separated from the boundary in $\mathcal{B}$. Indeed, let $X = w_U(0)$, for some $U \in \Gamma$. For each $x \in K$, the vector $Xx \oplus x$ can be written as $U(0 \oplus y)$ where $y \in K$. Hence $\|Xx \oplus x\|^2 \leq C^2 \|y\|^2$ and $[Xx \oplus x, Xx \oplus x] = [0 \oplus y, 0 \oplus y]$. Thus $\|Xx\|^2 - \|x\|^2 = -\|y\|^2$ and $\|Xx\|^2 + \|x\|^2 \leq C^2 \|y\|^2$. It follows that $\|Xx\|^2 \leq r^2 \|x\|^2$ where $r = \sqrt{\frac{C^2-1}{C^2+1}} < 1$. Thus $\|X\| \leq r$ for all $X \in G(0)$.

By Corollary 5.1, G has a fixed point $X \in \mathcal{B}$. Hence the maximal negative space $N = S(X)$ is invariant under Γ. Denoting by M the $[\cdot, \cdot]$-orthogonal complement of N it is easy to see that (M, N) is an invariant dual pair of subspaces. $\qquad\square$

Now we turn to the problem of orthogonalization of a representation of a group on a Hilbert space (the standard reference of the topic is the book [18]). As for representations in complex Hilbert spaces (see [17]), we say that a representation π of a group Γ in a real Hilbert space $\mathcal{H}$ preserves a sesquilinear form with finite number of negative squares if $\mathcal{H}$ can be decomposed into the orthogonal sums of subspaces H and K (with $\dim K < \infty$) in such a way that $(P_H \pi(g)x, \pi(g)y) - (P_K \pi(g)x, \pi(g)y) = (P_H x, y) - (P_K x, y)$ for all $x, y \in \mathcal{H}$ and all $g \in G$.

Corollary 5.3. *If a bounded representation of a group Γ by operators in a real Hilbert space $\mathcal{H}$ preserves a sesquilinear form with finite number of negative squares, then it is similar to an orthogonal representation of Γ on $\mathcal{H}$.*

Proof. By our assumptions, π acts by J-unitary operators on a Π_k space $H \oplus K$. Since $\Gamma_0 = \pi(\Gamma)$ is a bounded group of J-unitary operators, the previous Corollary shows that there is an invariant for Γ_0 dual pair (M, N) of subspaces. Moreover $N = UK$ for some J-unitary operator U. Then all operators $\tau(g) = U^{-1}\pi(g)U$ are J-unitary, and the subspace K is invariant for them. It follows that H is also invariant for operators $\tau(g)$. Hence these operators preserve the scalar product on $\mathcal{H}$. Thus $g \mapsto \tau(g)$ is a unitary representation similar to π. $\qquad\square$

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